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A Characterization of Smoothness in Terms of Approximation by Algebraic Polynomials in L_p

VLADIMIR A. OPERSTEIN*

Department of Mathematics and Statistics, Simon Fraser University, Burnaby, British Columbia, V5A 1S6, Canada

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We prove direct and inverse theorems for the classical modulus of smoothness and approximation by algebraic polynomials in $L_p[-1, 1]$. These theorems contain the well-known theorems of A. Timan, V. Dzyadyk, G. Freud, and Yu. Brudnyi as special cases when $p=\infty$. They also provide a characterization of the spaces Lip(α, p) (Lipschitz spaces in L_p) for $0<\alpha<\infty$, $1 \leqslant p \leqslant \infty$.

1. INTRODUCTION

In this paper we study a connection between smoothness of functions in $L_p[-1, 1]$ and their approximability by algebraic polynomials. The smoothness is expressed in terms of the classical L_p-modulus of smoothness

$$\omega_r(f, t)_p := \sup_{0 \leqslant h \leqslant t} \left\| \sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} f(\cdot + ih) \right\|_{L_p[-1, 1-rh]}.$$

Such a connection in the case $p=\infty$ was described as far back as in the 1960s. But for $p<\infty$ only partial results have been obtained to date. The theorems presented in this paper make the situation in the case $1 \leqslant p < \infty$ as complete as in the case $p=\infty$.

Let us recall the general theorems which express the connection between $\omega_r(f, t)_p$ and approximation by polynomials in the case $p=\infty$:

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THEOREM A [1]. *Let f be a continuous function given on the interval $[-1, 1]$. For every $r = 1, 2, \ldots$ there exists a sequence of algebraic polynomials P_n of degree at most n, $n \geqslant r-1$, such that*

$$|f(x) - P_n(x)| \leqslant c\omega_r(f, \rho_n(x))_\infty, \tag{1.1}$$

where $\rho_n(x) = n^{-1}(1-x^2)^{1/2} + n^{-2}$ and c is a constant depending only on r.

THEOREM B [9]. *Let f be a function defined on $[-1, 1]$. If, for some sequence of algebraic polynomials P_n of degree at most n, $n = 1, 2, \ldots$,*

$$|f(x) - P_n(x)| \leqslant \omega(\rho_n(x)), \tag{1.2}$$

and $\omega(t)$ is a nondecreasing function satisfying the condition $\omega(t_1 + t_2) \leqslant M[\omega(t_1) + \omega(t_2)]$, then for every $r = 1, 2, \ldots$,

$$\omega_r(f, t)_\infty \leqslant ct^r \int_t^1 \frac{\omega(u)}{u^{r+1}}\, du \qquad \left(0 < t \leqslant \frac{1}{2}\right), \tag{1.3}$$

where the constant c depends only on r and M.

An essential feature of these theorems is that the rate of approximation is faster at the end points than at interior points of the interval. Thus (in contrast to corresponding theorems for trigonometric polynomials [7]) the results do not have simple formulations in terms of supremum norms.

Theorems A and B complement each other in the following sense: Denote by $\mathrm{Lip}(\alpha, p) = \{f \in L_p[-1, 1] : \omega_r(f, t)_p = O(t^\alpha)\}$, $0 < \alpha < r$, the Lipschitz spaces in $L_p[-1, 1]$. The following characterization of $\mathrm{Lip}(\alpha, \infty)$ in terms of approximation is an immediate corollary of Theorems A and B.

THEOREM C. *A function f, defined on $[-1, 1]$, belongs to $\mathrm{Lip}(\alpha, \infty)$ if and only if there exist polynomials P_n of degree at most n, $n = 1, 2, \ldots$, such that*

$$|f(x) - P_n(x)| \leqslant c\rho_n(x)^\alpha. \tag{1.4}$$

The problem of extending Theorems A, B, and C to the case $p < \infty$ has been open for a long time. We offer the following theorems as a solution to this problem.

2. STATEMENT OF THE RESULTS

Throughout the rest of the paper ω will denote a nondecreasing function $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the condition $\omega(t_1 + t_2) \leqslant M[\omega(t_1) + \omega(t_2)]$ for some constant M. From this point on ρ_k will denote the function $\rho_k(x) = 2^{-k}(1-x^2)^{1/2} + 2^{-2k}$. In the theorems stated below the l_p norm is taken over the discrete parameter $k = 0, 1, 2, \ldots$; the L_p norm is taken over a continuous parameter $x \in [-1, 1]$, though dependence on x of the functions f, ρ_k, and P_k is not explicitly indicated. We use the customary notation for the mixed norm $\|A_k(\cdot)\|_{l_p(L_p)} := \|\{\|A_k(\cdot)\|_{L_p}\}_k\|_{l_p}$.

THEOREM 1. *Let $f \in L_p[-1, 1]$. For every $r = 1, 2, \ldots$ there exists a sequence of algebraic polynomials P_k of degree at most $2^k + r - 2$, $k = 0, 1, 2, \ldots$, such that*

$$\left\|\frac{f - P_k}{\omega(\rho_k)}\right\|_{l_p(L_p)} \leqslant c \cdot \left\|\frac{\omega_r(f, 2^{-k})_p}{\omega(2^{-k})}\right\|_{l_p}, \tag{2.1}$$

where the constant c depends on r and M, but on nothing else.

Theorem A follows from Theorem 1 by setting $\omega(t) = \omega_r(f, t)_p$ and $p = \infty$.

Remark. The inequality (2.1) with the weight function $\omega(t) = t^\alpha$ is due to E. Dyn'kin [5]. However, when restricted to this case, the inequality (2.1) does not contain (1.1) and furthermore the functions $\omega(t) = t^\alpha$ cannot be used to characterize $\mathrm{Lip}(\alpha, p)$ when $p < \infty$ (cf. [5]).

THEOREM 2. *Let f be a function defined on $[-1, 1]$. If for some sequence of algebraic polynomials P_k of degree not exceeding $2^k - 1$*

$$\left\|\frac{f - P_k}{\omega(\rho_k)}\right\|_{l_p(L_p)} \leqslant 1, \tag{2.2}$$

then, for every $r = 1, 2, \ldots$,

$$\omega_r(f, t)_p \leqslant ct^r \left[\int_t^1 \left(\frac{\omega(u)}{u^r}\right)^q \frac{du}{u}\right]^{1/q}, \qquad \frac{1}{p} + \frac{1}{q} = 1, \tag{2.3}$$

where the constant c depends only on r and M, but on nothing else.

Setting $p = \infty$, we obtain Theorem B as a special case of Theorem 2.

Theorems 1 and 2 provide the following characterization of the Lipschitz spaces $\mathrm{Lip}(\alpha, p)$:

THEOREM 3. *A function f, defined on $[-1, 1]$, belongs to* $\mathrm{Lip}(\alpha, p)$ *if and only if there exist polynomials P_k of degree at most 2^k, $k = 0, 1, 2, \ldots$, such that*

$$\|(f - P_k) \min\{1, t/\rho_k\}^s\|_{l_p(L_p)} = O(t^\alpha), \qquad 0 < \alpha < s. \tag{2.4}$$

Remarks. (i) The idea of using $\min\{1, t/\rho_n\}$ for a characterization of $\mathrm{Lip}(\alpha, p)$ appears in [4], where it is proved that for every function f from the Lipschitz class $\mathrm{Lip}(\alpha, p)$, $0<\alpha<1$, there exist polynomials P_k such that $\|(f-P_k)\min\{1, t/\rho_k\}\|_{l_\infty(L_p)} = O(t^\alpha)$.

(ii) It has been shown by V. Motornyi [8] and R. DeVore [4] that the direct L_p-analog of (1.4), namely the condition $\|(f-P_n)\,\rho_n^{-\alpha}\|_{L_p} \leqslant c$, does not characterize $\mathrm{Lip}(\alpha, p)$ when $p<\infty$. The proofs in Section 3 show that the mixed norm $l_p(L_p)$ appears naturally and seems to be inevitable.

3. Proofs

Notation. For a set $J \subseteq [-1, 1]$ we use the following notations: the Lebesgue measure of J is denoted by $|J|$, χ_J is the characteristic function of the set J, and $L_p(J)$ denotes the L_p norm taken over J.

We will write $A \ll B$ if there is a constant $c \geqslant 1$ such that $A \leqslant c\cdot B$, and use the notation and $A \sim B$ if $c^{-1}\cdot A \leqslant B \leqslant c\cdot A$; in both cases the constant c may depend on r and M, but on nothing else. We also remark that the usual change for sup should be made in formulas when $p=\infty$ or $q=\infty$.

Proof of Theorem 1. We will derive the inequality (2.1) from the following two properties of algebraic polynomials:

(i) Approximation of characteristic functions of intervals [2]. For every $n = 1, 2, \ldots$, $m = 1, 2, \ldots$, and $y \in [-1, 1]$ there exists a polynomial p of degree at most $n-1$ such that

$$|\chi_{[y,1]}(x) - p(x)| \leqslant c(1 + n\,|\arccos x - \arccos y|)^{-m}. \tag{3.1}$$

Here $x \in [-1, 1]$ and the constant c depends only on m.

(ii) A growth condition for polynomials. For every polynomial q of degree not exceeding n and every interval $I = [a, b]$,

$$|q(x)| \leqslant c_n\,\mathrm{dist}(x, I)^n \frac{\|q\|_{Lp(I)}}{|I|^{n+1/p}}, \qquad x \notin I, \tag{3.2}$$

where $\mathrm{dist}(x, I) = |x-(a+b)/2|$.

The inequality (3.2) follows immediately from [9, Formulae 2.9.11(9), 2.9.1(5), and Section 4.9.6].

We shall apply properties (i) and (ii) on certain subintervals of $[-1, 1]$. We use the partitions

$$x_{k,j} = \cos(\pi - 2^{-k-1}\pi(j-1)), \qquad j = 1, \ldots, 2^{k+1}+1, \qquad k = 0, 1, \ldots,$$

which are consistent with (i). For each $k=0, 1, 2, \ldots$ we denote the intervals

$$I_{k,j}=[x_{k,j}, x_{k,j+1}),\ j=1, \ldots, 2^{k+1}-1; \qquad I_{k,2^{k+1}}=[x_{k,2^{k+1}}, 1],$$

and numbers $\omega_{k,j}=\omega(|I_{k,j}|)$, $j=1, \ldots, 2^{k+1}$. For each $l=0, 1, 2, \ldots$ let $\mathscr{B}_l=\{(k, j): 2^{-l-1}<|I_{k,j}|\leqslant 2^{-l}\}$.

LEMMA. *For every* $k=0, 1, 2, \ldots,$ *every* $i, j=1, \ldots, 2^{k+1}$, *and every* $l=0, 1, 2, \ldots$

$$\text{(i)}\quad |I_{k,i}|\ll(1+|i-j|)\,|I_{k,j}|, \tag{3.3}$$

$$\text{(ii)}\quad |I_{k,i}|\sim\rho_k(x) \qquad \textit{for all}\quad x\in I_{k,i}, \tag{3.4}$$

$$\text{(iii)}\quad \sum_{\mathscr{B}_l}\chi_{I_{k,i}}\leqslant c, \tag{3.5}$$

where c *is an absolute constant.*

Proof. This is a straightforward direct calculation.

We will also use the intervals $\bar{I}_{k,j}=I_{k,j}\cup I_{k,j+1}$, $k=0, 1, 2, \ldots,$ $j=1, \ldots, 2^{k+1}-1$, and the notation $\bar{\omega}_{k,j}=\omega(|\bar{I}_{k,j}|)$. For each $l=0, 1, 2, \ldots$ let $\bar{\mathscr{B}}_l=\{(k,j): 2^{-l-1}<|\bar{I}_{k,j}|\leqslant 2^{-l}\}$. The lemma also clearly holds for $\bar{I}_{k,j}$ and $\bar{\mathscr{B}}_l$.

Let $p_{k,j}$ be polynomials of degree at most 2^k-1 for $k=0, 1, 2, \ldots$ and $j=1, \ldots, 2^{k+1}$. Suppose that they satisfy (3.1) with $y=x_{k,j}$ and $m=2r+[\alpha]+1$, where $\alpha=\log_2(2M)$. It follows from the lemma that for every $k=0, 1, 2, \ldots,$ every $i, j=1, \ldots, 2^{k+1}$, and every $x\in I_{k,i}$

$$|\chi_{k,j}(x)-p_{k,j}(x)|\ll(1+|i-j|)^{-m}, \tag{3.6}$$

where $\chi_{k,j}=\chi_{[x_{k,j}, 1]}$.

It also follows from (3.2) and the lemma that

$$|q(x)|\ll(1+|i-j|)^{2(r-1)+1/p}\cdot|I_{k,i}|^{-1/p}\cdot\|q\|_{L_p(I_{k,j})}, \tag{3.7}$$

for any polynomial q of degree not exceeding $r-1$ and every $x\in I_{k,i}$.

The monotonicity properties of ω imply that $\omega(ct)\leqslant 2Mc^{\alpha}\omega(t)$, for $c>1$ and $\alpha=\log_2(2M)$. Therefore, by the lemma, for every $k=0, 1, 2, \ldots,$ every $i, j=1, \ldots, 2^{k+1}$, and every $x\in I_{k,i}$

$$\omega(\rho_k(x))^{-1}\ll(1+|i-j|)^{\alpha}\omega_{k,j}^{-1}. \tag{3.8}$$

The estimates (3.6), (3.7), and (3.8) imply that for any polynomial q of degree not exceeding $r-1$, for every $k=0, 1, 2, \ldots,$ every $i, j=1, \ldots, 2^{k+1}$, and every $x\in I_{k,i}$

$$\left|\frac{q(x)(\chi_{k,j}(x)-p_{k,j}(x))}{\omega(\rho_k(x))}\right|$$
$$\ll (1+|i-j|)^{-2}\,|I_{k,i}|^{-1/p}\cdot\frac{\|q\|_{L_p(I_{k,j})}}{\omega_{k,j}}. \tag{3.9}$$

It follows from (3.9) that for arbitrary polynomials q_j of degree at most $r-1$, $j=1, ..., 2^{k+1}$,

$$\left\|\sum_{j=1}^{2^{k+1}} q_j\frac{\chi_{k,j}-p_{k,j}}{\omega(\rho_k)}\right\|_{L_p}\ll\left[\sum_{j=1}^{2^{k+1}}\left(\frac{\|q_j\|_{L_p(I_{k,j})}}{\omega_{k,j}}\right)^p\right]^{1/p}. \tag{3.10}$$

Let $L_{k,j}$ be arbitrary polynomials of degree at most $r-1$ for each $k=0, 1, ...$ and $j=1, ..., 2^{k+1}-1$. Define $q_{k,j}=L_{k,j}-L_{k,j-1}$ when $j=2, ..., 2^{k+1}-1$ and $q_{k,j}=0$ when $j=1$ or 2^{k+1}. By (3.3)

$$\left[\sum_{j=1}^{2^{k+1}}\left(\frac{\|q_{k,j}\|_{L_p(I_{k,j})}}{\omega_{k,j}}\right)^p\right]^{1/p}\ll\left[\sum_{j=1}^{2^{k+1}-1}\left(\frac{\|f-L_{k,j}\|_{L_p(\bar I_{k,j})}}{\bar\omega_{k,j}}\right)^p\right]^{1/p}. \tag{3.11}$$

Define $S_k = L_{k,1}+\sum_{j=2}^{2^{k+1}-1} q_{k,j}\chi_{k,j}$ and $P_k = L_{k,1}+\sum_{j=2}^{2^{k+1}-1} q_{k,j}p_{k,j}$. Clearly S_k is a piecewise-polynomial function and P_k is a polynomial of degree not exceeding 2^k+r-2. By (3.10),

$$\left\|\frac{S_k-P_k}{\omega(\rho_k)}\right\|_{L_p}\ll\left[\sum_{j=1}^{2^{k+1}-1}\left(\frac{\|q_{k,j}\|_{L_p(I_{k,j})}}{\omega_{k,j}}\right)^p\right]^{1/p}.$$

Therefore,

$$\left\|\frac{f-P_k}{\omega(\rho_k)}\right\|_{L_p}\ll\left[\sum_{j=1}^{2^{k+1}-1}\left(\frac{\|f-L_{k,j}\|_{L_p(\bar I_{k,j})}}{\bar\omega_{k,j}}\right)^p\right]^{1/p}.$$

Since

$$\left[\sum_{k=0}^{\infty}\sum_{j=1}^{2^{k+1}-1}\left(\frac{\|f-L_{k,j}\|_{L_p(\bar I_{k,j})}}{\bar\omega_{k,j}}\right)^p\right]^{1/p}$$
$$\ll\left[\sum_{l=0}^{\infty}\sum_{\mathscr{B}_l}\left(\frac{\|f-L_{k,j}\|_{L_p(\bar I_{k,j})}}{\omega(2^{-l})}\right)^p\right]^{1/p},$$

we obtain

$$\left\|\frac{f-P_k}{\omega(\rho_k)}\right\|_{l_p(L_p)}\ll\left[\sum_{l=0}^{\infty}\sum_{\mathscr{B}_l}\left(\frac{\|f-L_{k,j}\|_{L_p(\bar I_{k,j})}}{\omega(2^{-l})}\right)^p\right]^{1/p}. \tag{3.12}$$

To complete the proof we have to specialize to the case where the polynomials $L_{k,j}$ of degree at most $r-1$ are chosen so that they satisfy the condition

$$\left[\sum_{\mathscr{B}_l} \|f - L_{k,j}\|_{L_p(\bar{I}_{k,j})}^p\right]^{1/p} \ll \omega_r(f, 2^{-l})_p. \tag{3.13}$$

We obtain such polynomials in the following way: For each $l = 0, 1, 2, \ldots$ there is a function f_l satisfying the condition (see [3])

$$\|f - f_l\|_{L_p} + 2^{-lr} \|f_l^{(r)}\|_{L_p} \ll \omega_r(f, 2^{-l})_p. \tag{3.14}$$

For every $k = 0, 1, 2, \ldots$ and $j = 1, \ldots, 2^{k+1} - 1$ we define polynomials $L_{k,j}$ by the formula $L_{k,j} = \sum_{s=0}^{r-1} c_s(x - x_{k,j})^s$, where $c_s = f_l^{(s)}(x_{k,j})/(s!)$ and $l = [\log_2(1/|\bar{I}_{k,j}|)]$. Using part (iii) of the lemma applied to the intervals $\bar{I}_{k,j}$ and the sets of indices $\mathscr{B}_l$, and also the estimate (3.14) and Taylor's formula, we obtain (3.13).

It follows from (3.12) and (3.13) that

$$\left\|\frac{f - P_k}{\omega(\rho_k)}\right\|_{l_p(L_p)} \ll \left\|\frac{\omega_r(f, 2^{-k})_p}{\omega(2^{-k})}\right\|_{l_p}, \tag{3.15}$$

which completes the proof of Theorem 1. ∎

Proof of Theorem 2. In view of the monotonicity properties of ω and $\omega_r(f, t)_p$ it suffices to give a proof for $t = 2^{-m}$ where $m = 2, 4, \ldots$. Let m be an arbitrary positive even integer fixed throughout the proof.

We let $I = [-1, 1 - r2^{-m}]$. For every $n = 0, 1, \ldots, m/2$ we define the intervals I_n by setting

$$I_n = \{x \in I: 2^{-1} < 2^n(1 - x^2)^{1/2} \leqslant 1\} \qquad \text{if} \quad n = 0, 1, \ldots, m/2 - 1$$

and

$$I_n = \{x \in I: 2^{m/2}(1 - x^2)^{1/2} \leqslant 1\} \qquad \text{if} \quad n = m/2.$$

We also define the sets $I_n^{\pm} = \{x \in I_n: \pm x \geqslant 0\}$. For a set J it will be convenient to use the notation $\bar{J} = \{x + t: x \in J, 0 \leqslant t \leqslant r \cdot 2^{-m}\}$, for example, $\bar{I}_n$, $\bar{I}_n^{\pm}$, etc. We observe that $\sum_{n=0}^{m/2} \chi_{\bar{I}_n} \leqslant C$ where C is a constant depending only on r. The sets $\bar{I}_n$ are "consistent" with $\rho_k(x)$ in the sense that for every $n = 0, \ldots, m/2$ and $x \in \bar{I}_n$

$$\rho_{m-n}(x) \sim 2^{-m}, \tag{3.16}$$

where the constant depends only on r.

LEMMA. *Let* $1 \leqslant q \leqslant \infty$. *For every* $n = 0, 1, \ldots, m/2$ *and each* $x \in \bar{I}_n$,

$$\left[\sum_{k=0}^{m-n} \left(\frac{\omega(\rho_k(x))}{\rho_k(x)^r}\right)^q\right]^{1/q} \ll \left[\sum_{k=0}^{m} \left(\frac{\omega(2^{-k})}{2^{-kr}}\right)^q\right]^{1/q}. \tag{3.17}$$

Proof. Let $\delta_{k,n} = 2^{-k} + 2^{-2(k-n)}$ for each $k = 0, 1, 2, \ldots$ and $n = 0, 1, \ldots, m/2$. The monotonicity properties of ω and the equivalent relation (3.16) imply that for every $x \in \bar{I}_n$

$$\left[\sum_{k=0}^{m-n} \left(\frac{\omega(\rho_k(x))}{\rho_k(x)^r}\right)^q\right]^{1/q} \ll \left[\sum_{k=n}^{m} \left(\frac{\omega(\delta_{k,n})}{\delta_{k,n}^r}\right)^q\right]^{1/q}.$$

When $k \leqslant 2n$, the inequalities $\delta_{k,n} \leqslant 2^{-2(k-n)+1}$ and $2n \leqslant m$ yield

$$\left[\sum_{k=n}^{2n} \left(\frac{\omega(\delta_{k,n})}{\delta_{k,n}^r}\right)^q\right]^{1/q} \ll \left[\sum_{k=0}^{m} \left(\frac{\omega(2^{-k})}{2^{-kr}}\right)^q\right]^{1/q}.$$

When $2n < k \leqslant m$, the inequality $\delta_{k,n} \leqslant 2^{-k+1}$ yields

$$\left[\sum_{k=2n+1}^{m} \left(\frac{\omega(\delta_{k,n})}{\delta_{k,n}^r}\right)^q\right]^{1/q} \ll \left[\sum_{k=2n+1}^{m} \left(\frac{\omega(2^{-k})}{2^{-kr}}\right)^q\right]^{1/q}.$$

These estimates imply (3.17). ∎

For $J \subset I$ and $0 \leqslant h \leqslant 2^{-m}$ we will use the estimate $\|\Delta_h^r g\|_{L_p(J)} \leqslant c_r \|g\|_{L_p(\bar{J})}$ if $g \in L_p[-1, 1]$, and the estimate $\|\Delta_h^r g\|_{L_p(J)} \leqslant 2^{-mr} \|g^{(r)}\|_{L_p(\bar{J})}$ if $g^{(r)} \in L_p[-1, 1]$.

Let us now suppose that $\{P_k\}_{k=0}^{\infty}$ is any sequence of polynomials satisfying the hypotheses of Theorem 2. For every $h \in [0, 2^{-m}]$

$$\|\Delta_h^r f\|_{L_p(I)} \leqslant \left[\sum_{n=0}^{m/2} \|f - P_{m-n}\|_{L_p(\bar{I}_n)}^p\right]^{1/p} + 2^{-mr} \left[\sum_{n=0}^{m/2} \|P_{m-n}^{(r)}\|_{L_p(\bar{I}_n)}^p\right]^{1/p}. \tag{3.18}$$

It follows from (3.16) that

$$\|f - P_{m-n}\|_{L_p(\bar{I}_n)} \asymp \|(f - P_{m-n})\, \omega(\rho_{m-n})^{-1}\|_{L_p(\bar{I}_n)} \cdot \omega(2^{-m}). \tag{3.19}$$

Let $p^{-1} + q^{-1} = 1$. It follows from (3.19) that

$$\left[\sum_{n=0}^{m/2} \|f - P_{m-n}\|_{L_p(\bar{I}_n)}^p\right]^{1/p} \leqslant 2^{-mr} \left[\sum_{k=0}^{m} \left(\frac{\omega(2^{-k})}{2^{-kr}}\right)^q\right]^{1/q} \cdot \left\|\frac{f - P_k}{\omega(\rho_k)}\right\|_{l_p(L_p)}. \tag{3.20}$$

We shall now use telescoping sums to estimate the second term of the sum in the inequality (3.18). Let $Q_k = P_k - P_{k-1}$ when $k = 1, ..., m$, and $Q_0 = 0$. Hölder's inequality and the lemma imply that for every $x \in [-1, 1]$

$$\left|\sum_{k=0}^{m-n} Q_k^{(r)}(x)\right| \leqslant \left[\sum_{k=0}^{m-n} \left(\frac{Q_k^{(r)}(x)\, \rho_k^r(x)}{\omega(\rho_k(x))}\right)^p\right]^{1/p} \cdot \left[\sum_{k=0}^{m} \left(\frac{\omega(2^{-k})}{2^{-kr}}\right)^q\right]^{1/q}.$$

Consequently,

$$\left[\sum_{n=0}^{m/2} \|P_{m-n}^{(r)}\|_{L_p(\bar{I}_n)}\right]^{1/p} \ll \left\|\frac{Q_k^{(r)}\rho_k^r}{\omega(\rho_k)}\right\|_{l_p(L_p)} \cdot \left[\sum_{k=0}^{m} \left(\frac{\omega(2^{-k})}{2^{-kr}}\right)^q\right]^{1/q}. \tag{3.21}$$

The following inequality of the Markov–Bernstein type (see [6]) holds for any polynomial Q_k of degree not exceeding 2^k:

$$\left\|\frac{Q_k^{(r)}\rho_k^r}{\omega(\rho_k)}\right\|_{L_p} \ll \left\|\frac{Q_k}{\omega(\rho_k)}\right\|_{L_p}. \tag{3.22}$$

Using the inequalities (3.21), (3.22), and the fact that $\omega(\rho_k) \sim \omega(\rho_{k-1})$, we obtain the estimate

$$\left[\sum_{n=0}^{m/2} \|P_{m-n}^{(r)}\|_{L_p(\bar{I}_n)}^p\right]^{1/p} \ll \left\|\frac{f-P_k}{\omega(\rho_k)}\right\|_{l_p(L_p)} \cdot \left[\sum_{k=0}^{m} \left(\frac{\omega(2^{-k})}{2^{-kr}}\right)^q\right]^{1/q}. \tag{3.23}$$

The estimates (3.18), (3.20), and (3.23) imply that

$$\omega_r(f, 2^{-m})_p \ll \left\|\frac{f-P_k}{\omega(\rho_k)}\right\|_{l_p(L_p)} \cdot 2^{-mr} \left[\sum_{k=0}^{m} \left(\frac{\omega(2^{-k})}{2^{-kr}}\right)^q\right]^{1/q}. \tag{3.24}$$

The inequality (2.3) immediately follows from (3.24), (2.2), and the monotonicity properties of ω. ∎

Proof of Theorem 3. Necessity follows from Theorem 1 by choosing $\omega(u) = \max\{1, u/t\}^s$. Sufficiency follows from Theorem 2 by choosing $\omega(u) = t^\alpha \max\{1, u/t\}^s$ and $r > s$. ∎

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